

Calculus III: Coursework 6
M.A.H. MacCallum 2006

The deadline is 5PM on Tuesday November 14th. Hand-in is to the green box in the basement of the Maths building.

Note: The period for this work includes the test week, Week 7. You will need to plan how to divide your time spent on this course over the next two weeks between this coursework and revision for the mid-term test.

1. Using the identities for differentiation of products,

- (a) Evaluate the divergence and curl of $\mathbf{a}(\mathbf{r} \cdot \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors. [You may find it useful to work out $\nabla(\mathbf{b} \cdot \mathbf{r})$ first.]
- (b) Prove that if $\mathbf{A}$ and $\mathbf{B}$ are irrotational, then $\mathbf{A} \times \mathbf{B}$ is solenoidal.
- (c) Prove that for any function $f(r)$, $f(r)\mathbf{r}$ is irrotational.
- (d) Prove that $(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v})$.

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Answer

- (a) $\nabla(\mathbf{r} \cdot \mathbf{b}) = \mathbf{r} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{r}) + (\mathbf{r} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{r} = (\mathbf{b} \cdot \nabla)\mathbf{r} = \mathbf{b}$
using the facts that all derivatives of $\mathbf{a}$ and $\mathbf{b}$ vanish, $\nabla \times \mathbf{r} = \mathbf{0}$, and $(\mathbf{b} \cdot \nabla)\mathbf{r} = \mathbf{b}$. The last two of these were proved in lectures. Hence
 $\nabla \cdot (\mathbf{a}(\mathbf{r} \cdot \mathbf{b})) = \mathbf{a} \cdot \nabla(\mathbf{r} \cdot \mathbf{b}) + (\mathbf{r} \cdot \mathbf{b})(\nabla \cdot \mathbf{a}) = \mathbf{a} \cdot \mathbf{b}$ and
 $\nabla \times (\mathbf{a}(\mathbf{r} \cdot \mathbf{b})) = (\mathbf{r} \cdot \mathbf{b})\nabla \times \mathbf{a} - \mathbf{a} \times \nabla(\mathbf{r} \cdot \mathbf{b}) = -\mathbf{a} \times \mathbf{b}$
- (b) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} = 0$.
- (c)

$$\begin{aligned}\nabla \times (f(r)\mathbf{r}) &= f(r)\nabla \times \mathbf{r} - \mathbf{r} \times \nabla f \\ &= -\mathbf{r} \times \nabla f \\ &= -\mathbf{r} \times \frac{\partial f}{\partial r} \nabla r \\ &= -\frac{\partial f}{\partial r}(\mathbf{r} \times \mathbf{r})/r = \mathbf{0}\end{aligned}$$

- (d) $\nabla(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = 2(\mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v})$
and rearrange.

Marks: Part (a), 5 for each part plus 5 for $\nabla(\mathbf{r} \cdot \mathbf{b})$ wherever it is calculated (in each bit, 2 for choosing a correct identity, 3 for accuracy). (b) 5, (c) 4 for first identity and accurate use, 3 for chain rule use, 3 for the rest, (d) 4 for using the correct identity, 6 for accurate completion.

2. Write the following in ordinary vector notation.

- (i) $a_k b_k$, (ii) $\delta_{ij} a_j$, (iii) $\epsilon_{ijk} a_j b_k$, (iv) δ_{ii} , (v) $\partial f / \partial x_i$,
(vi) $a_k b_p c_m d_k c_p - \delta_{mj} b_j = \epsilon_{mij} f_i d_j$.

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Answers

- (i) $\mathbf{a} \cdot \mathbf{b}$
- (ii) $\mathbf{a}$
- (iii) $\mathbf{a} \times \mathbf{b}$
- (iv) 3
- (v) ∇f
- (vi) $(\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})\mathbf{c} - \mathbf{b} = \mathbf{f} \times \mathbf{d}$

Marks 2 each for first 5 parts (all seen in lectures). 8 for last part (not seen), of which 4 for spotting the disguised dot products.

3. Write the vector equation

$$\mathbf{a} = \mathbf{b} \times \mathbf{c} - (\mathbf{b} \cdot \mathbf{d})\mathbf{f}$$

in index notation.

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Answer $a_i = \epsilon_{ijk}b_jc_k - (b_md_m)f_i$

Marks: 5 for an expression with a set of indices obeying the conventions, 5 for accuracy.

4. Re-do question 1 using index notation.

[12]

Answer

x_i is the position vector $\mathbf{r}$, so

$$\nabla \cdot (\mathbf{a}(\mathbf{r} \cdot \mathbf{b})) = \partial_i(a_ib_jx_j) = a_ib_j\delta_{ij} = a_ib_i = \mathbf{a} \cdot \mathbf{b}$$

$$\nabla \times (\mathbf{a}(\mathbf{r} \cdot \mathbf{b}))_i = \epsilon_{ijk}\partial_j(a_kb_mx_m) = \epsilon_{ijk}a_kb_m\delta_{jm} = \epsilon_{ijk}a_kb_j = (\mathbf{b} \times \mathbf{a})_i$$

Marks: 5 for div and 7 for curl (dealing with epsilon is harder than with just sums)

5. Using (without proof) the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$,

- (a) find an expression not involving cross-products for $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$;
- (b) and prove that $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - \mathbf{G}(\nabla \cdot \mathbf{F}) - (\mathbf{F} \cdot \nabla)\mathbf{G}$,
where by definition $(\mathbf{a} \cdot \nabla)\mathbf{b}$ means the vector whose i th component is $a_j\partial b_i/\partial x_j$;
and
- (c) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$.

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Answer

(a) $\epsilon_{ijk}a_jb_k\epsilon_{ilm}c_ld_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})a_jb_kc_ld_m = a_lc_lb_md_m - a_md_mb_kc_k$

so $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$

(b) $\epsilon_{ijk}\partial_j(\epsilon_{klm}F_lG_m) = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j(F_lG_m) = \partial_m(F_iG_m) - \partial_l(F_lG_i) = F_i(\partial_mG_m) + G_m\partial_mF_i - G_i(\partial_mF_m) - F_m\partial_mG_i$

so $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - \mathbf{G}(\nabla \cdot \mathbf{F}) - (\mathbf{F} \cdot \nabla)\mathbf{G}$

(c) $\epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_lF_m) = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_lF_m = \partial_i\partial_mF_m - \partial_l\partial_lF_i$ so

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}.$$

Marks: 7 each for parts (a) and (b), 6 for part (c).